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On the finite temperature formalism in integrable quantum field theories

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Abstract
Two different theoretical formulations of the finite temperature effects have been recently proposed for integrable field theories. In order to decide which of them is the correct one, we perform for a particular model an explicit check of their predictions for the one-point function of the trace of the stress–energy tensor, a quantity which can be independently determined by the thermodynamical Bethe ansatz.

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1. Introduction

Finite temperature correlation functions are important quantities for many applications of both theoretical and experimental interest (see, for instance [1]). A special class of quantum field theories is provided by the two-dimensional integrable models, which can be exactly solved by means of bootstrap methods [2–5]. For these models, two different formulations of finite temperature effects have been recently discussed in the literature: the first is due to LeClair and Mussardo [6] and the second has been proposed by Delfino [7]. Although the two formalisms coincide if applied to the trivial cases of free quantum field theories, however, they drastically differ once used to deal with interacting theories. To determine which of the two is the correct one we compare their predictions versus a quantity which can be independently determined. This is the case of the finite temperature one-point function of the trace of the stress–energy tensor which can be computed by the thermodynamical Bethe ansatz (TBA) [5]. As we will show below, the proposal by LeClair and Mussardo exactly matches the low-temperature expansion of this quantity whereas the proposal by Delfino fails at order $\mathcal{O}(\mathrm{e}^{-3mr})$. Before presenting the explicit calculations, let us briefly discuss the main features of the two different finite temperature formalisms.

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2. LeClair–Mussardo formalism

This formalism, discussed in [6], combines together physical principles coming from two different areas: the thermodynamical Bethe ansatz and the form factor approach. It originates from an interpretation of the expression of the free energy—as determined by the TBA—in terms of quasi-particle excitations with respect to a thermal ground state. In order to clarify this statement, it is useful to summarize the TBA approach. We assume for simplicity that the spectrum of the integrable theory consists of a single particle A with mass m and an exact S-matrix $S(\theta)$. In the following we consider the case $S(0) = -1$, which gives rise to the fermionic TBA equations. We define

$$\sigma(\theta) = -\mathrm{i}\,\log S(\theta) \qquad \phi(\theta) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log S(\theta). \tag{2.1}$$

The partition function at a finite temperature T and on a volume L (for $L \to \infty$) is determined by means of the thermodynamical Bethe ansatz equations as follows [5]. In a box of large volume L, $0 < x < L$, with periodic boundary conditions, the quantization condition of the momenta is given by $\mathrm{e}^{\mathrm{i}k(\theta_i)L}\prod_{j\neq i} S(\theta_i - \theta_j) = 1$, i.e.

$$mL\,\mathrm{sh}\theta_i + \sum_{j\neq i}\sigma(\theta_i - \theta_j) = 2\pi n_i \tag{2.2}$$

where n_i are integers. Introducing a density of occupied states per unit volume $\rho_1(\theta)$ as well as a density of levels $\rho(\theta)$, in the thermodynamic limit equation (2.2) becomes

$$2\pi\rho = e + 2\pi\phi * \rho_1 \tag{2.3}$$

where $e = m\,\cosh\theta$ and $(f*g)(\theta) = \int_{-\infty}^{\infty}\mathrm{d}\theta' f(\theta-\theta')g(\theta')/2\pi$. Defining the pseudo-energy $\varepsilon(\theta)$ as

$$\frac{\rho_1}{\rho} = \frac{1}{1+\mathrm{e}^{\varepsilon}} \tag{2.4}$$

the minimization of the free energy with respect to the densities of states leads to the integral equation

$$\varepsilon = eR - \phi * \log(1+\mathrm{e}^{-\varepsilon}) \tag{2.5}$$

and the partition function is then given by

$$Z(L,R) = \exp\left[mL\int\frac{\mathrm{d}\theta}{2\pi}\mathrm{ch}\,\theta\,\log\!\left(1+\mathrm{e}^{-\varepsilon(\theta)}\right)\right]. \tag{2.6}$$

As shown in [6], the interesting point is now that the above partition function can be interpreted as the one of a gas of fermionic particles but with energy given by $\varepsilon(\theta)/R$. Namely, there is a one-to-one correspondance between the above expression (2.6) and the partition function computed according to the following thermal sum

$$Z(L,R) = \sum_{n=0}^{\infty}\frac{1}{n!}\int\frac{\mathrm{d}\theta_1}{2\pi}\cdots\frac{\mathrm{d}\theta_n}{2\pi}\langle\theta_n\cdots\theta_1|\theta_1\cdots\theta_n\rangle\prod_{i=1}^{n}\mathrm{e}^{-\varepsilon(\theta_i)} \tag{2.7}$$

where the scalar products of the states are computed by applying the standard free fermionic rules. The above equality implies that all physical properties of the system can be extracted by employing the quasi-particle excitations above the TBA thermal ground state. Since this differs from the usual (zero temperature) ground state, it is not surprising that its excitations do not satisfy the standard dispersion relations $e = m\cosh\beta$, $p = m\sinh\beta$, rather they have dressed energy $\tilde{e} = \varepsilon(\theta)/R$ and dressed momentum $\tilde{k}(\theta)$:

$$\tilde{e}(\theta) = \varepsilon(\theta)/R \qquad \tilde{k}(\theta) = k(\theta) + 2\pi(\sigma * \rho_1)(\theta). \tag{2.8}$$

In this context, the rapidity θ plays the role of a variable which simply parametrizes the dispersion relation of the quasi-particle excitations and their S-matrix, which is assumed to coincide with the original $S(\theta_i - \theta_j)$.

The TBA allows us to compute the finite temperature one-point function of the trace of the stress–energy tensor T_μ^μ [5]. In fact, we have

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = \frac{2\pi}{R} \frac{\mathrm{d}}{\mathrm{d}R}[RE(R)], \tag{2.9}$$

where $E(R) = -\log Z/L$. This can be also expressed as

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = m \int \mathrm{d}\theta \frac{\mathrm{e}^{-\varepsilon}}{1+\mathrm{e}^{-\varepsilon}} \left(\partial_R \varepsilon \,\mathrm{ch}\theta - \frac{1}{R} \partial_\theta \varepsilon \,\mathrm{sh}\theta \right) \tag{2.10}$$

where the functions $\partial_R \varepsilon$ and $\partial_\theta \varepsilon$ satisfy linear integral equations which can be easily solved. The final result reads

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = 2\pi m^2 \left(\sum_{n=1}^{\infty} \int \left[\prod_{i=1}^{n} \frac{\mathrm{d}\theta_i}{2\pi} f(\theta_i) \mathrm{e}^{-\varepsilon(\theta_i)} \right] \phi(\theta_{12}) \cdots \phi(\theta_{n-1,n}) \,\mathrm{ch}(\theta_{1n}) \right) \tag{2.11}$$

where

$$f(\theta) = \frac{1}{1+\mathrm{e}^{-\varepsilon(\theta)}}. \tag{2.12}$$

Let us now consider the calculation of the finite temperature one-point functions (the only ones which we consider in this paper). According to LeClair and Mussardo, this correlator is given by

$$\langle \mathcal{O}(x,t) \rangle_R = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(2\pi)^n} \int \left[\prod_{i=1}^{n} \mathrm{d}\theta_i f(\theta_i) \mathrm{e}^{-\varepsilon(\theta_i)} \right] \langle \theta_n \cdots \theta_1 | \mathcal{O}(0) | \theta_1 \cdots \theta_n \rangle_{\mathrm{conn}} \tag{2.13}$$

where the connected form factor of the operator $\mathcal{O}$ is defined as

$$\langle \theta_n \cdots \theta_1 | \mathcal{O} | \theta_1' \cdots \theta_m' \rangle_{\mathrm{conn}} \equiv \mathrm{FP} \left(\lim_{\eta_i \to 0} \langle 0 | \mathcal{O} | \theta_n + \mathrm{i}\pi + \mathrm{i}\eta_n, \ldots, \theta_1 + \mathrm{i}\pi + \mathrm{i}\eta_1, \theta_1, \ldots, \theta_n \rangle \right). \tag{2.14}$$

FP in front of the above expression means taking its *finite part*, i.e. terms proportional to $(1/\eta_i)^p$, where p is some positive power, and also terms proportional to η_i/η_j, $i \neq j$ are discarded in taking the limit. With this prescription the resulting expression is a universal quantity, i.e. independent of the way in which the above limits are taken.

It is easy to see that within this formalism, the finite temperature one-point function of the trace of the stress–energy tensor exactly coincides with its expression provided by the TBA, equation (2.11). In fact, the connected matrix elements of this operators are given by

$$\begin{aligned} \langle \theta | T_\mu^\mu | \theta \rangle_{\mathrm{conn}} &= 2\pi m^2 \\ \langle \theta_2, \theta_1 | T_\mu^\mu | \theta_1, \theta_2 \rangle_{\mathrm{conn}} &= 4\pi m^2 \phi(\theta_1 - \theta_2) \,\mathrm{ch}(\theta_1 - \theta_2) \end{aligned} \tag{2.15}$$

and by an inductive application of the form factor residue equations

$$\langle \theta_n \cdots \theta_1 | T_\mu^\mu | \theta_1 \cdots \theta_n \rangle_{\mathrm{conn}} = 2\pi m^2 \phi(\theta_{12}) \phi(\theta_{23}) \cdots \phi(\theta_{n-1,n}) \,\mathrm{ch}(\theta_{1n}) + \text{permutations} \tag{2.16}$$

where $\theta_{ij} = \theta_i - \theta_j$. Once inserted into equation (2.13), the above series coincides with that of equation (2.11).

In conclusion, the formalism by LeClair and Mussardo predicts, at least for the particular thermal one-point function of T_μ^μ, an exact matching with the expression determined by the TBA.

3. Delfino's formalism

This formalism, discussed in [7], only employs the form factor approach. The finite temperature effects are taken into account by defining the theory on a cylinder infinitely extended in the space direction and a width $R = 1/T$ in the other direction. The particles entering the thermal sum are the asymptotic states satisfying the standard dispersion relations $e = m\cosh\beta$, $p = m\sinh\beta$ and the contribution of the n-particle asymptotic state to $\mathrm{Tr}[\mathcal{O}\mathrm{e}^{-HR}]$ is given by

$$d_n^{\mathcal{O}}(R) = \frac{1}{n!}\frac{1}{(2\pi)^n}\int \mathrm{d}\theta_1\ldots\mathrm{d}\theta_n F_{n,n}^{\mathcal{O}}(\theta_n,\ldots,\theta_1|\theta_1,\cdots,\theta_n)\mathrm{e}^{-E_nR} \tag{3.1}$$

with $E_n = m\sum_{i=1}^n \cosh\theta_i$ and

$$F_{m,n}^{\mathcal{O}}(\theta_m',\ldots,\theta_1'|\theta_1,\ldots,\theta_n) = \langle\theta_m',\ldots,\theta_1'|\mathcal{O}|\theta_1,\ldots,\theta_n\rangle.$$

Define

$$d^{\mathcal{O}}(R) = \sum_{n=0}^{\infty} d_n^{\mathcal{O}}(R) \tag{3.2}$$

and normalize the thermal sum with respect to the identity operator I

$$\langle\mathcal{O}\rangle_R = \frac{d^{\mathcal{O}}(R)}{d^I(R)}. \tag{3.3}$$

In his paper [7], Delfino considered for the finite part of the form factors entering equation (3.1) the *symmetric limit*

$$\mathcal{F}_{2n}^{\mathcal{O}}(\theta_1,\ldots,\theta_n) = \lim_{\eta\to 0} F_{0,2n}^{\mathcal{O}}(\theta_n + \mathrm{i}\pi + \mathrm{i}\eta,\ldots,\theta_1 + \mathrm{i}\pi + \mathrm{i}\eta,\theta_1,\ldots,\theta_n) \tag{3.4}$$

and he also showed that the singular disconned parts of the form factors of the local operator $\mathcal{O}$ only enter through the constant factor $S(0)$. All other singular terms cancel in the ratio (3.3). Finally, he proposed for the finite temperature one-point function the expression

$$\langle\mathcal{O}\rangle_R = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(2\pi)^n}\int\left[\prod_{i=1}^n \mathrm{d}\theta_i\, g(\theta_i,R)\mathrm{e}^{-mR\,\cosh\theta_i}\right]\mathcal{F}_{2n}^{\mathcal{O}}(\theta_1,\ldots,\theta_n) \tag{3.5}$$

where

$$g(\theta,R) = \frac{1}{1 - S(0)\mathrm{e}^{-mR\,\cosh\theta}}. \tag{3.6}$$

The above formula has to be contrasted with the one given by equation (2.13).

4. Main differences and open problems

There are two main differences between the two formalisms:

- LeClair–Mussardo formalism employs the quasi-particle excitations with respect to the thermal vacuum and therefore the pseudo-energy $\varepsilon(\theta)$, solution of the integral equation (2.5), whereas Delfino's formalism employs the standard asymptotic particles at zero temperature with energy $e = m\,\cosh\theta$ and momentum $p = m\,\sinh\theta$. These different choices of excitations seem somehow related to the boundary conditions adopted by the two formalisms along the space direction, i.e. in the LeClair–Mussardo approach one considers a box of large volume L, with periodic b.c., in the limit $L \to \infty$, whereas in the Delfino approach one directly considers the infinitely extended line. Note, however, that there is no dependence on L in the final expressions (2.13) and (3.5) and therefore the role played by the boundary conditions in thermal effects and which of the two is the appropriate one is not *a priori* clear.

- The form factors entering equation (2.13) are computed according to the prescription given by equation (2.14) whereas those entering equation (3.5) are computed according to the symmetric limit (3.4). The two different prescriptions for the finite part of the form factors produce, of course, two different results. In the case of the trace of the stress–energy tensor, for instance, there is already a difference for the two-particle form factor entering the thermal sum: by using the symmetric limit, in fact we have

$$\left\langle \theta_2, \theta_1 \left| T_\mu^\mu \right| \theta_1, \theta_2 \right\rangle = 8\pi m^2 \phi(\theta_1 - \theta_2)\, \cosh^2 \frac{\theta_1 - \theta_2}{2} \tag{4.1}$$

to be contrasted with equation (2.15), obtained by using the other prescription.

It is therefore evident that the two formulas, equations (2.13) and (3.5), proposed for the one-point function at finite temperature, deeply differ their physical justifications and their technical details. To judge which of the two is the correct one it seems necessary to reach a better understanding of the physical principles ruling the thermal effects in quantum field theories. Given the present ignorance about these principles, it is therefore difficult to decide *a priori* in favour of one or the other of the two formulations and the best thing one can do is to perform some checks. Those already done and discussed in the literature are unfortunately inconclusive. Lukyanov [8], for instance, computed the thermal one-point functions of the vertex operators in the Sinh–Gordon model by performing the path integral of the model and he showed that these quantities coincide with the ones computed in the formalism by LeClair–Mussardo. Unfortunately, the perturbative order at which he performed the computation does not permit to decide about their general validity. On the other hand, Delfino [7] showed that his formalism is able to reproduce the one-point function of T_μ^μ up to the two-particle contributions but unfortunately he did not prove the complete equivalence of his formula with the TBA expression.

Given the present unsatisfactory status about the validity of the two formalisms it is highly desirable to perform additional checks, in particular by comparing their predictions against a quantity which can be determined by an independent method. These considerations naturally select the one-point function of the trace of the stress–energy tensor as a check quantity for the two formulas, since its expression (2.11) is independently determined by the TBA. Hence, we have to see whether or not Delfino's formula reproduces the TBA result, not only up to the two-particle contribution, but also to higher orders (as shown above, the formula by LeClair–Mussardo coincides with the formula of the TBA). We then have two possibilities: (i) the formula proposed by Delfino is unable to reproduce the TBA result at higher orders; (ii) the formula proposed by Delfino reproduces the TBA result, albiet it is just a different organization of the terms entering both the thermal sum and the integral equations of the TBA. In the first case, the failure of this check is already enough to decide about the general validity of the thermal expressions proposed by Delfino. In the second case, there would still be open the problem regarding which of the two formalisms is the correct one, since their coincidence for the particular case of the stress–energy tensor is not expected to occur for other operators. Luckily enough, it is the first possibility that happens. To show the discrepancy of Delfino's formula with the TBA, we compare the thermal expression of the stress–energy tensor of a particular model which can be analytically solved.

5. A simplified model

The main technical difficulty in comparing Delfino's expression of $\langle T_\mu^\mu \rangle_R$ with the analogous expression coming from the TBA lies in solving the integral equation (2.5). We can simplify this step by taking a local kernel, i.e. we consider an integrable model for which

$$\phi(\theta_1 - \theta_2) = 2\pi\delta(\theta_1 - \theta_2). \tag{5.1}$$

For the associate S-matrix we have

$$S(\theta) = \begin{cases} 1 & \text{if } \theta \neq 0 \\ -1 & \text{if } \theta = 0 \end{cases} \tag{5.2}$$

i.e., if we parametrize the S-matrix as $S(\theta) \equiv -\mathrm{e}^{i\sigma(\theta)}$, for the phase shift we have

$$\sigma(\theta) = \begin{cases} \pi & \text{if } \theta > 0 \\ 0 & \text{if } \theta = 0 \\ -\pi & \text{if } \theta < 0. \end{cases} \tag{5.3}$$

The legitimacy of the above S-matrix is discussed in the appendix and it is based on the observation that the integrable model defined in this way may be regarded as the limit $g \to 0$ of the Sinh–Gordon model. In fact, with the notation of reference [9], the S-matrix of the Sinh–Gordon model is given by

$$S_{\mathrm{Sh}}(\theta) = \frac{\sinh\theta - \mathrm{i}\sin\frac{\pi B(g)}{2}}{\sinh\theta + \mathrm{i}\sin\frac{\pi B(g)}{2}} \tag{5.4}$$

with $B(g) = 2g^2/8\pi + g^2$ and g the coupling constant of the model (see equation (A.5)). It is convenient to define $B(g) \equiv 2\alpha$. For the corresponding kernel we have

$$\phi_{\mathrm{Sh}}(\theta) = \frac{2\sin\pi\alpha\cosh\theta}{\sinh^2\theta + \sin^2\pi\alpha} \tag{5.5}$$

and in the limit $\alpha \to 0$ we have

$$\lim_{\alpha\to 0}\phi_{\mathrm{Sh}}(\theta) = 2\pi\delta(\theta). \tag{5.6}$$

By using the kernel (5.1), the integral equation (2.5) becomes

$$\varepsilon(\theta) = mR\cosh\theta - \ln\left(1 + \mathrm{e}^{-\varepsilon(\theta)}\right) \tag{5.7}$$

whose solution is given by

$$\varepsilon(\theta) = \ln(\mathrm{e}^{mR\cosh\theta} - 1). \tag{5.8}$$

Hence

$$f(\theta)\,\mathrm{e}^{-\varepsilon(\theta)} = \frac{\mathrm{e}^{-\varepsilon}}{1 + \mathrm{e}^{-\varepsilon}} = \mathrm{e}^{-mR\cosh\theta} \tag{5.9}$$

and inserting into the TBA formula (2.11), we have

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = 2\pi m^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} [\mathrm{e}^{-mR\cosh\theta} + \mathrm{e}^{-2mR\cosh\theta} + \mathrm{e}^{-3mR\cosh\theta} + \cdots]. \tag{5.10}$$

For the purpose of comparing with Delfino's prediction, it is convenient to explicitly leave the n-particle contributions to the thermal average, although it is evident that the above series can summed to

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = 2\pi m^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} \frac{1}{\mathrm{e}^{mR\cosh\theta} - 1} \tag{5.11}$$

which is nothing else but the thermal one-point function of T_μ^μ for a free bosonic theory (see the appendix for further details on this issue).

Let us now consider the form factors of T_μ^μ associated with the simplified model with kernel (5.1). In virtue of the observed equivalence of this theory with a particular limit of the Sinh–Gordon model, the form factors can be obtained by a careful $g \to 0$ limit of the corresponding quantities of the Sinh–Gordon model. They were computed in [9] and can be expressed as

$$\left\langle 0 \left| T_\mu^\mu(0) \right| \theta_1, \ldots, \theta_n \right\rangle = \frac{2\pi m^2}{F_{\min}(\mathrm{i}\pi)} \left(\frac{4 \sin \pi\alpha}{F_{\min}(\mathrm{i}\pi)} \right)^{n-1} \mathcal{Q}_n(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\min}(\theta_{ij})}{x_i + x_j}. \tag{5.12}$$

A few words on the above expression: the explicit form of $F_{\min}(\theta)$ can be found in [9]. For our purposes we only need the functional equation satisfied by $F_{\min}(\theta)$

$$F_{\min}(\theta) F_{\min}(\theta + \mathrm{i}\pi) = \frac{\sinh\theta}{\sinh\theta + \mathrm{i}\sin\pi\alpha}. \tag{5.13}$$

$\mathcal{Q}_n$ is a symmetric polynomial in the variables $x_i \equiv \mathrm{e}^{\theta_i}$ given by

$$\mathcal{Q}_n(x_1, \ldots, x_n) = \det M_{ij}$$

with the $(n-1) \times (n-1)$ matrix M_{ij} given by

$$M_{ij} = \sigma_{2i-j}[i - j + 1].$$

In the above equation the symbol $[n]$ is defined by

$$[n] \equiv \frac{\sin(n\alpha)}{\sin\alpha}$$

and σ_k is the elementary symmetric polynomial given by the generating function

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k(x_1, x_2, \ldots, x_n).$$

In the limit $\alpha \to 0$, the first polynomials $\mathcal{Q}_n$ are given by

$$\mathcal{Q}_2 = \sigma_1 \qquad \mathcal{Q}_4 = \sigma_1\sigma_2\sigma_3 \qquad \mathcal{Q}_6 = \sigma_1\sigma_5\left[\sigma_2\sigma_3\sigma_4 + 3\sigma_3\sigma_6 - 4(\sigma_1\sigma_2\sigma_6 + \sigma_4\sigma_5)\right]. \tag{5.14}$$

5.1. Two-particle contribution

By using equation (5.12), let us compute

$$\left\langle \theta_2, \theta_1 \left| T_\mu^\mu \right| \theta_1, \theta_2 \right\rangle = \lim_{\eta_1 \to 0} \lim_{\eta_2 \to 0} \left\langle 0 \left| T_\mu^\mu \right| \theta_1 + \mathrm{i}\pi + \eta_1, \theta_2 + \mathrm{i}\pi + \eta_2, \theta_1, \theta_2 \right\rangle. \tag{5.15}$$

We will consider the contributions coming from the different terms in (5.12) separately.

By using the functional equation (5.13), for the product of $F_{\min}(\theta_{ij})$ we have, in the above limit,

$$\prod_{i<j} F_{\min}(\theta_{ij}) \longrightarrow [F_{\min}(i\pi)]^2 \frac{\sinh^2\theta_{12}}{\sinh^2\theta_{12} + \sin^2\pi\alpha}. \tag{5.16}$$

For the polynomial of the denominator we have

$$\prod_{i<j}(x_i + x_j) \longrightarrow A_1 A_2 x_1 x_2 (x_1 + x_2)^2 (x_1 - x_2)^2 \tag{5.17}$$

where $A_k = (1 - \mathrm{e}^{\mathrm{i}\eta_k}) \sim -i\eta_k$. Finally, for the polynomial $\mathcal{Q}_4$ in the numerator we obtain

$$\mathcal{Q}_4 \longrightarrow x_1 x_2 \left(x_1^2 + x_2^2\right) \left[\left(A_1^2 + A_2^2\right) x_1 x_2 + A_1 A_2 \left(x_1^2 + x_2^2\right) \right]. \tag{5.18}$$

We now have two possibilities. The first consists of keeping in the above expression only the term multiplying the combination $A_1 A_2$ (and disregarding those multiplying $(A_1^2 + A_2^2)$). This leads to the computation of the connected form factor. In this case, combining all terms and taking the limit (5.15), we have

$$\langle \theta_2, \theta_1 | T_\mu^\mu | \theta_1, \theta_2 \rangle_{\mathrm{conn}} = 4\pi m^2 \left(\frac{2 \sin \pi\alpha \cosh \theta_{12}}{\sinh^2 \theta_{12} + \sin^2 \pi\alpha} \right) \cosh \theta_{12}. \tag{5.19}$$

Now by taking the limit $\alpha \to 0$ and using equation (5.6), we have

$$\langle \theta_2, \theta_1 | T_\mu^\mu | \theta_1, \theta_2 \rangle_{\mathrm{conn}} = 4\pi m^2 \phi(\theta_1 - \theta_2) \cosh \theta_{12} \tag{5.20}$$

in agreement with equation (2.15).

The second possibility consists of taking the symmetric limit considered by Delfino. This is obtained by taking $A_1 = A_2$. In this case, the symmetric limit of equation (5.15) produces

$$\langle \theta_2, \theta_1 | T_\mu^\mu | \theta_1, \theta_2 \rangle_{\mathrm{sym}} = 8\pi m^2 \left(\frac{2 \sin \pi\alpha \cosh \theta_{12}}{\sinh^2 \theta_{12} + \sin^2 \pi\alpha} \right) \cosh^2 \frac{\theta_{12}}{2}. \tag{5.21}$$

Now by taking the limit $\alpha \to 0$ and using equation (5.6), we have

$$\langle \theta_2, \theta_1 | T_\mu^\mu | \theta_1, \theta_2 \rangle_{\mathrm{sym}} = 8\pi m^2 \phi(\theta_1 - \theta_2) \cosh^2 \frac{\theta_{12}}{2}. \tag{5.22}$$

Let us consider the expression (3.5) up to the the two-particle contribution. For the function $g(\theta, R)$ we have

$$g(\theta, R) = \frac{1}{1 - S(0)\mathrm{e}^{-mR\cosh\theta}} = \frac{1}{1 + \mathrm{e}^{-mR\cosh\theta}} \tag{5.23}$$

and then

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = 2\pi m^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} \left[\frac{\mathrm{e}^{-mR\cosh\theta}}{1 + \mathrm{e}^{-mR\cosh\theta}} + 2 \frac{\mathrm{e}^{-2mR\cosh\theta}}{(1 + \mathrm{e}^{-mR\cosh\theta})^2} + \cdots \right]. \tag{5.24}$$

Expanding this expression in power of $\mathrm{e}^{-mR\cosh\theta}$ up to $\mathrm{e}^{-2mR\cosh\theta}$ we have

$$\langle T_\mu^\mu \rangle_R - \langle T_\mu^\mu \rangle_0 = 2\pi m^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} [\mathrm{e}^{-mR\cosh\theta} + \mathrm{e}^{-2mR\cosh\theta} + \mathcal{O}(\mathrm{e}^{-3mR\cosh\theta})]. \tag{5.25}$$

Now comparing this expression with equation (5.9), we explicitly confirm the agreement found at this order by Delfino in his paper.

5.2. Three-particle contribution

By using equation (5.12), let us compute

$$\begin{aligned} \langle \theta_3, \theta_2, \theta_1 | T_\mu^\mu | \theta_1, \theta_2, \theta_2 \rangle = \lim_{\eta_1 \to 0} \lim_{\eta_2 \to 0} \lim_{\eta_3 \to 0} \langle 0 | T_\mu^\mu | \theta_1 + i\pi \\ + \eta_1, \theta_2 + \mathrm{i}\pi + \eta_2, \theta_3 + \mathrm{i}\pi + \eta_3, \theta_1, \theta_2, \theta_3 \rangle. \end{aligned} \tag{5.26}$$

As before, let us consider the contributions coming from different terms separately. By using the functional equation (5.13), for the product of $F_{\min}(\theta_{ij})$ we have, in the above limit,

$$\begin{aligned} \prod_{i<j} F_{\min}(\theta_{ij}) \longrightarrow [F_{\min}(\mathrm{i}\pi)]^3 \left(\frac{\sinh^2 \theta_{12}}{\sinh^2 \theta_{12} + \sin^2 \pi\alpha} \right) \\ \times \left(\frac{\sinh^2 \theta_{13}}{\sinh^2 \theta_{13} + \sin^2 \pi\alpha} \right) \left(\frac{\sinh^2 \theta_{23}}{\sinh^2 \theta_{23} + \sin^2 \pi\alpha} \right). \end{aligned} \tag{5.27}$$

For the polynomial of the denominator we have

$$\prod_{i<j}(x_i + x_j) \longrightarrow A_1A_2A_3x_1x_2x_3 \left[\left(x_1^2 - x_2^2\right)\left(x_1^2 - x_3^2\right)\left(x_2^2 - x_3^2\right)\right]^2$$
$$= 64\, A_1A_2A_3(x_1x_2x_3)^5(\sinh\theta_{12}\sinh\theta_{13}\sinh\theta_{23})^2. \tag{5.28}$$

For the polynomial $\mathcal{Q}_6$, we have two possibilities. The first consists of keeping only the term multiplying the combination $A_1A_2A_3$ (and disregarding all other expressions which multiply the other monomials like A_1^3, $A_1^2A_2$, etc). This leads to the computation of the connected form factor. In this case we have

$$\mathcal{Q}_6^{\text{conn}} \longrightarrow A_1A_2A_3x_1x_2x_3\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_3^2\right)\left(x_2^2 + x_3^2\right)\Big[(x_1x_2)^2\left(x_1^2 + x_2^2 - 2x_3^2\right)$$
$$+ (x_1x_3)^2\left(x_1^2 + x_3^2 - 2x_2^2\right) + (x_2x_3)^2\left(x_2^2 + x_3^2 - 2x_1^2\right)\Big] \tag{5.29}$$

and for the connected form factor, combining all terms, we obtain

$$\left\langle\theta_3, \theta_2, \theta_1\left|T_\mu^\mu\right|\theta_1, \theta_2, \theta_2\right\rangle_{\text{conn}} = 2\pi m^2\left(\frac{2\sin\pi\alpha\cosh\theta_{12}}{\sinh^2\theta_{12} + \sin^2\pi\alpha}\right)\left(\frac{2\sin\pi\alpha\cosh\theta_{23}}{\sinh^2\theta_{23} + \sin^2\pi\alpha}\right)$$
$$\times\frac{\sinh^2\theta_{13}}{\sinh^2\theta_{13} + \sin^2\pi\alpha}\cosh\theta_{13} + \text{permutations}. \tag{5.30}$$

By taking the limit $\alpha \to 0$, we obtain the result reported in formula (2.16).

The second possibility consists of considering the symmetric limit, which is obtained by taking $A \equiv A_1 = A_2 = A_3$. Other terms enter the polynomial in this case and we have

$$\mathcal{Q}_6^{\text{sym}} \longrightarrow A^3x_1x_2x_3(x_1 + x_2 + x_3)(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1x_2 + x_1x_3 + x_2x_3)$$
$$\times\Big[(x_1x_2)^2\left(x_1^2 + x_2^2 - 2x_3^2\right) + (x_1x_3)^2\left(x_1^2 + x_3^2 - 2x_2^2\right)$$
$$+ (x_2x_3)^2\left(x_2^2 + x_3^2 - 2x_1^2\right)$$
$$- 2x_1x_2x_3\left[x_2(x_1 - x_3)^2 + x_1(x_2 - x_3)^2 + x_3(x_1 - x_2)^2\right]\Big]. \tag{5.31}$$

Therefore, combining all the different contributions, in the symmetric limit we have

$$\left\langle\theta_3, \theta_2, \theta_1\left|T_\mu^\mu\right|\theta_1, \theta_2, \theta_3\right\rangle_{\text{symm}} = 2\pi m^2\left(\frac{2\sin\pi\alpha\cosh\theta_{12}}{\sinh^2\theta_{12} + \sin^2\pi\alpha}\right)\left(\frac{2\sin\pi\alpha\cosh\theta_{13}}{\sinh^2\theta_{13} + \sin^2\pi\alpha}\right)$$
$$\times\frac{\sinh^2\theta_{23}}{\sinh^2\theta_{23} + \sin^2\pi\alpha}[2(\cosh\theta_{12} + \cosh\theta_{13} + \cosh\theta_{23}) + 3]$$
$$\times\frac{\cosh\frac{\theta_{12}}{2}\cosh\frac{\theta_{13}}{2}}{\cosh\theta_{12}\cosh\theta_{13}}\frac{\left(2\cosh^2\frac{\theta_{23}}{2} - 1\right)}{\cosh\frac{\theta_{23}}{2}} + \text{permutations}.$$

By taking now the limit $\alpha \to 0$ and using equation (5.6), we obtain

$$\left\langle\theta_3, \theta_2, \theta_1\left|T_\mu^\mu\right|\theta_1, \theta_2, \theta_2\right\rangle_{\text{sym}} = 2\pi m^2\phi(\theta_1 - \theta_2)\phi(\theta_1 - \theta_3)\frac{\cosh\frac{\theta_{12}}{2}\cosh\frac{\theta_{13}}{2}}{\cosh\theta_{12}\cosh\theta_{13}}$$
$$\times[2(\cosh\theta_{12} + \cosh\theta_{13} + \cosh\theta_{23}) + 3]\frac{\left(2\cosh^2\frac{\theta_{23}}{2} - 1\right)}{\cosh\frac{\theta_{23}}{2}}$$
$$+ \text{permutations}. \tag{5.32}$$

Once inserted into equation (3.5), we have

$$\langle T_\mu^\mu\rangle_R - \langle T_\mu^\mu\rangle_0 = 2\pi m^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} \left[\frac{\mathrm{e}^{-mR\cosh\theta}}{1+\mathrm{e}^{-mR\cosh\theta}} \right. \\ \left. + 2\frac{\mathrm{e}^{-2mR\cosh\theta}}{(1+\mathrm{e}^{-mR\cosh\theta})^2} + \frac{9}{2}\frac{\mathrm{e}^{-3mR\cosh\theta}}{(1+\mathrm{e}^{-mR\cosh\theta})^3} + \cdots \right] \tag{5.33}$$

and by making an expansion up to $\mathrm{e}^{-3mR\cosh\theta}$ we have

$$\langle T_\mu^\mu\rangle_R - \langle T_\mu^\mu\rangle_0 = 2\pi m^2 \int_{-\infty}^{+\infty} \frac{\mathrm{d}\theta}{2\pi} \left[\mathrm{e}^{-mR\cosh\theta} + \mathrm{e}^{-2mR\cosh\theta} + \frac{3}{2}\mathrm{e}^{-3mR\cosh\theta} + \mathcal{O}(\mathrm{e}^{-4mR}) \right] \tag{5.34}$$

i.e. the third order coefficient disagrees with the corresponding coefficient of equation (5.10).

6. Conclusions

In this paper we have critically analysed the status of the thermal formalism for two-dimensional integrable field theory by comparing the approach proposed by LeClair and Mussardo with the approach proposed by Delfino. Whereas the first approach is able to reproduce the one-point function of T_μ^μ as given by the TBA, the second one is in agreement with the TBA formula only up to the two-particle contribution and differs otherwise. This has been explicitly shown by considering a simple integrable model, where all calculations can be performed analytically without relying on the solution of integral equation. It would be useful to further explore the subject and see whether or not the approach by LeClair and Mussardo passes other tests.

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Appendix A.

Consider the exact S-matrix of the Sinh–Gordon model

$$S_{\mathrm{Sh}}(\theta) = \frac{\sinh\theta - \mathrm{i}\sin\pi\alpha}{\sinh\theta + \mathrm{i}\sin\pi\alpha} \tag{A.1}$$

and study the above function in the limit $\alpha \to 0$. By choosing $\alpha = 1/n$, define the sequence of functions

$$S_n(\theta) = \frac{\sinh\theta - \mathrm{i}\sin\frac{\pi}{n}}{\sinh\theta + \mathrm{i}\sin\frac{\pi}{n}}. \tag{A.2}$$

These functions satisfy the following conditions for any value of n

$$S_n(\mathrm{i}\pi - \theta) = S_n(\theta) \qquad S_n(0) = -1. \tag{A.3}$$

The first condition expresses the crossing invariance of these functions and the second the fact that any element of the sequence corresponds to a fermionic S-matrix. The above functions can be equivalently expressed in terms of an integral representation as

$$S_n(\theta) = -\exp\left[2\mathrm{i}\int_0^\infty \frac{\mathrm{d}t}{t}\frac{\cosh\frac{t}{2}\left(1-\frac{2}{n}\right)}{\cosh\frac{t}{2}}\sin\frac{\theta t}{\pi}\right] \equiv -\exp[\mathrm{i}\sigma_n(\theta)]. \tag{A.4}$$

Subtleties arise in the limit $n \to \infty$. In fact, the naive way of taking the limit produces $S = 1$, a result which of course matches the physical intuition that the limit $g \to 0$ of the

Sinh–Gordon lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{g^2}(\cosh g\varphi - 1) \tag{A.5}$$

is the bosonic free-field theory. However, the proper way of taking the limit produces the discontinous phase shift

$$\sigma(\theta) = \lim_{n\to\infty} \sigma_n(\theta) = \begin{cases} \pi & \text{if } \theta > 0 \\ 0 & \text{if } \theta = 0 \\ -\pi & \text{if } \theta < 0 \end{cases} \tag{A.6}$$

as seen by equation (A.4). Equivalently, one can consider the limit of the logarithmic derivatives

$$\phi_n(\theta) = -\mathrm{i}\frac{1}{S_n}\frac{\mathrm{d}S_n(\theta)}{\mathrm{d}\theta} = \frac{2\sin\frac{\pi}{n}\cosh\theta}{\sinh^2\theta + \sin^2\frac{\pi}{n}}$$

i.e.

$$\lim_{n\to n} \phi_n(\theta) = 2\pi\delta(\theta) \tag{A.7}$$

and then integrate the above expression to obtain the phase shift. The limit $n \to \infty$ in equation (A.2) defines an S-matrix $S(\theta)$, which for real values of θ is given by

$$S(\theta) = \lim_{n\to\infty} S_n(\theta) = \begin{cases} 1 & \text{if } \theta \neq 0 \\ -1 & \text{if } \theta = 0 \end{cases} \tag{A.8}$$

By virtue of the first equation in (A.3), the above function satisfies the equation $S(\theta) = S(i\pi - \theta)$ and one can use this relation to extend it along the imaginary axis. Although $S(\theta)$ is a crossing symmetric function and a fermionic-type S-matrix, it is obviously discontinous and different from the naive limit $S = 1$. There is, however, no contradiction between the two results because all physical observables computed by using the two different S-matrices perfectly coincide. For instance, in order to compute the free energy of the theory, by using $S = 1$, one adopts the (free) TBA relative to a *bosonic-type* S-matrix, with the result

$$F(R) = \frac{R}{\pi}\int_0^\infty \mathrm{d}\theta \cosh\theta \ln(1 - \mathrm{e}^{-mR\cosh\theta}). \tag{A.9}$$

Conversely, by using the singular fermionic–type S matrix, with kernel (A.7), one arrives at the result

$$F(R) = -\frac{R}{\pi}\int_0^\infty \mathrm{d}\theta \cosh\theta \ln(1 + \mathrm{e}^{-\epsilon}) \tag{A.10}$$

where $\epsilon(\theta) = \ln(\mathrm{e}^{mR\cosh\theta} - 1)$ is the solution of the fermionic TBA equation (2.5). It is easy to see that, by inserting the expression $\epsilon(\theta)$in (A.10), the above two expressions of the free energy are equal. The same matching is also obtained by considering the calculation of any correlation function. In fact, for the free theory (the one with $S = 1$), this calculation can be done by using the Wick theorem. Obversely, for the theory defined through the singular fermionic-type S-matrix $S(\theta)$, one should employ the form factors $\langle 0|\mathcal{O}(0)|\theta_1, \ldots, \theta_n\rangle$ of the Sinh–Gordon theory, provided in reference [9], in the limit $\alpha \to 0$. These form factors, for real values of θ, however go uniformly to the corresponding form factors of the free theory.

In conclusion, the above arguments can be summarized by saying that the free bosonic theory can be equivalently seen as the consistent limit of interacting theories with a crossing symmetric fermionic-type S-matrix[1]. Moreover, even though the S-matrix obtained in the

[1] In order to appreciate this point, note that the vice versa is not true: free fermionic theory cannot be regarded as consistent limit of interacting bosonic-type S-matrix; see reference [10] for the problems presented by bosonic S-matrices.

limit is a singular function (so that one needs to generalize the usual principles of analiticity of the *S*-matrix theory), all observables determined by it precisely coincide with those obtained by using directly the free-theory formulation.

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